Expected Eccentricity of Massive Black Hole Mergers in the LISA Frequency Band

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ABSTRACT

Mergers between two massive black holes (both with $M \sim 10^{2-7}\, M_\odot$) would produce strong signals in the LISA detector, even out to significant redshifts. Here we show that for gravitational wave frequencies $f_{\rm GW} > 10^{-4}$ Hz, such binaries are likely to have low eccentricities.

1. Introduction

Stellar-mass black holes can form in binaries in relative isolation. Once such a compact binary is formed, it can therefore evolve purely by emission of gravitational radiation. Although no stellar-mass BH-BH binaries are currently known, the properties of NS-NS binaries suggest that perhaps some BH-BH binaries will have palpable eccentricity in the LISA frequency band (which we take to be $f_{\rm GW} > 10^{-4}$ Hz).

More massive black holes are likely to evolve differently. It is conceivable that in the very early universe (say $z > 10-20$), some of the first metal-free stars could form in binaries, although at this point this is not supported by high-resolution numerical simulations. If such stars contain several hundred solar masses each, then perhaps a massive black hole binary could form in the same way that stellar-mass binaries form in the current universe.

However, it seems likely that the majority of mergers between two massive black holes occur because the black holes form separately, then are brought together by dynamical processes. For example, black holes could form at the centers of two different dark matter halos, then when the halos merge during hierarchical structure assembly, the black holes could eventually be brought together. In the current universe, it has been suggested that sufficiently dense and massive young star clusters could undergo a process of runaway mergers of massive main-sequence stars, perhaps then forming a $\sim 10^3\, M_\odot$ black hole. Recent work (e.g., Gürkan, Fregeau, & Rasio 2005; Fregeau et al. 2006) suggests that if the initial stellar

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binary fraction in such clusters exceeds $\sim$10% (which seems inevitable) then more than one black hole in this mass range could form, probably separated by a few thousand AU and initially unbound relative to each other. Rough estimates then suggest that the massive black holes will sink to the center of the cluster and merge within a few million years. In another scenario, a stellar cluster could sink in the nucleus of a galaxy; if the cluster contains a massive black hole, it could eventually merge with the supermassive black hole in the galactic center.

The result is that mergers between massive black holes very probably involve substantial dynamical interactions as opposed to pure evolution by emission of gravitational radiation. These dynamical effects can change the eccentricity. On their own, they would tend to make the probability distribution "thermal", i.e., $P(e) = 2e$, which has a mean of $e \approx 0.7$. A bias towards higher eccentricities could happen if capture by emission of gravitational radiation happens preferentially at large e. The actual eccentricity is therefore determined by the competition between dynamical wandering and gravitational wave circularization.

2. Calculation

To get an estimate of how circular a binary will be, we can compare the time to reduce eccentricity via gravitational radiation with the time to interact with enough stars to change the eccentricity significantly. Obviously, high eccentricity is possible if there is a close hyperbolic orbit, but we consider the much more likely case that the evolution is gradual.

Consider a binary with component masses m_1 and m_2, which thus has total mass $M = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/M$. Suppose that its semimajor axis is a and eccentricity is e. The Peters equations give

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5}\frac{G^3 \mu M^2}{c^5 a^3 (1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \quad (1)$$

and

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{304}{15} e \frac{G^3 \mu M^2}{c^5 a^4 (1-e^2)^{5/2}} \left(1 + \frac{121}{304}e^2\right) \ . \quad (2)$$

The characteristic time to change the eccentricity is then

$$\begin{aligned} \tau_{\rm GW} = e/|de/dt| \ &\approx (15/304) c^5 a^4 (1-e^2)^{5/2} / (G^3 \mu M^2) \\ &\approx 8 \times 10^{17}\ {\rm yr} (M_\odot/\mu)(M_\odot/M)^2 (a/1\ {\rm AU})^4 (1-e^2)^{5/2} \ . \end{aligned} \quad (3)$$

Here we neglect the near-unity factor $(1 + 121e^2/304)$.

We can rewrite this in terms of gravitational wave frequency. Let us consider in particular the frequency emitted at pericenter. If the orbit is substantially eccentric, then the orbital frequency at that point will be approximately $\sqrt{2}$ times the circular frequency at that radius (because the speed is $\sqrt{2}$ times greater than a circular orbit). If we designate a maximum gravitational wave frequency $f_{\rm max}$ to be double the frequency at pericenter, then

$$f_{\rm max} \approx \frac{1}{\pi}\left[\frac{2GM}{(a(1-e))^3}\right]^{1/2} . \tag{4}$$

Therefore $a^4 = 7.5\times 10^{-5}{\rm AU}^4(M/10^3\,M_\odot)^{4/3}(f_{\rm max}/10^{-4}\ {\rm Hz})^{-8/3}(1-e)^{-4}$, and

$$\begin{aligned}\tau_{\rm GW} &\approx 6\times 10^4\ {\rm yr}(\mu/10^3\,M_\odot)^{-1}(M/10^3\,M_\odot)^{-2/3}(f_{\rm max}/10^{-4}\ {\rm Hz})^{-8/3}(1+e)^{5/2}(1-e)^{-3/2}\\ &\approx 3\times 10^5\ {\rm yr}(\mu/10^3\,M_\odot)^{-1}(M/10^3\,M_\odot)^{-2/3}(f_{\rm max}/10^{-4}\ {\rm Hz})^{-8/3}(1-e)^{-3/2}\end{aligned} \tag{5}$$

where in the last line we assume a relatively high eccentricity, so that $1+e\approx 2$.

To determine the timescale for dynamical eccentricity changes, we note that the orbital speed at a gravitational wave frequency of 10^{-4} Hz is rather high; roughly 3000 km s$^{-1}(M/10^3\,M_\odot)^{1/3}$ for a circular orbit, for example. This is much greater than the velocity dispersion σ in any realistic astrophysical environment. Therefore, the cross section Σ for a binary-single encounter with a closest approach r_p is

$$\Sigma = \pi r_p^2\left[1+2GM/(r_p\sigma^2)\right]\approx \pi r_p(2GM/\sigma^2)\ . \tag{6}$$

If we assume that $r_p < 2a$ is required for a significant interaction, then

$$\Sigma = 4\pi a(GM/\sigma^2)\approx 2\times 10^{30}\ {\rm cm}^2(M/10^3\,M_\odot)^{4/3}(f_{\rm max}/10^{-4}\ {\rm Hz})^{-2/3}(\sigma/10\ {\rm km\ s}^{-1})^{-2}(1-e)^{-1}\ . \tag{7}$$

The timescale for a *single* interaction is just $\tau_{\rm int}=1/(n\Sigma\sigma)$, where n is the number density of objects interacting with the binary. However, note that for the eccentricity or semimajor axis of the binary to be affected significantly, the binary must interact with approximately a mass $\sim\mu$. If the stars interacting with the binary have average mass $\langle m\rangle$, this means that the time required for dynamical interactions to change the eccentricity is $\tau_{\rm dyn}\approx(\mu/\langle m\rangle)\tau_{\rm int}$. This yields the product $n\langle m\rangle$, which is simply the average mass density ρ of things interacting with the binary, so only this quantity matters and we have

$$\begin{aligned}\tau_{\rm dyn} &\approx 5\times 10^8\ {\rm yr}(\rho/10^6\,M_\odot\ {\rm pc}^{-3})^{-1}(\mu/10^3\,M_\odot)(M/10^3\,M_\odot)^{-4/3}\\ &\quad\times(f_{\rm max}/10^{-4}\ {\rm Hz})^{2/3}(\sigma/10\ {\rm km\ s}^{-1})(1-e)\ .\end{aligned} \tag{8}$$

The ratio between the circularization time and the time for dynamical change is therefore

$$\begin{aligned}\tau_{\rm GW}/\tau_{\rm dyn} &\approx 6\times 10^{-4}(\mu/10^3\,M_\odot)^{-2}(M/10^3\,M_\odot)^{2/3}(\rho/10^6\,M_\odot\ {\rm pc}^{-3})\\ &\quad\times(\sigma/10\ {\rm km\ s}^{-1})^{-1}(f_{\rm max}/10^{-4}\ {\rm Hz})^{-10/3}(1-e)^{-5/2}\ .\end{aligned} \tag{9}$$

If $\tau_{\rm GW}/\tau_{\rm dyn} \ll 1$ we conclude that circularization dominates; if the ratio is the other way, then circularization is ineffective and the eccentricity will sample a thermal distribution. Therefore, comparable mass binaries with $M > 10^3\,M_\odot$ will circularize long before they enter the LISA band. A $10^2\,M_\odot$ black hole spiraling into a $10^6\,M_\odot$ black hole could have a significant eccentricity at 10^{-4} Hz, but by 10^{-3} Hz will have circularized, and the evolution at $f_{\rm max} > 10^{-4}$ Hz is likely to be dominated by gravitational radiation. Note also that a classic extreme mass ratio inspiral, with $M = 10^6\,M_\odot$ and $\mu = 10\,M_\odot$, could have a significant eccentricity if (as expected in galactic nuclei) the orbits come in from large distances so that $e \sim 1 - 10^{-4}$ or larger.